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Lacunary series expansions in hyperholomorphic $F_G^{\alpha}(p,q,s)$ spaces

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Abstract: In this paper we define a new class of hyperholomorphic functions, which is known as $F_G^{\alpha}(p,q,s)$ spaces. We characterize hyperholomorphic functions in $F_G^{\alpha}(p,q,s)$ space in terms of the Hadamard gap in Quaternion analysis.

Keywords: Quaternionic analysis, $F_G^{\alpha}(p,q,s)$ spaces, lacunary Series.

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1. Introduction

Quaternions were introduced for the first time by William Rowan Hamilton in 1843 [1].The generalizations of the theory of holomorphic functions in one complex variable is known as Quaternion analysis [2–5]. Quaternions are also recognized as a powerful tool for modeling and solving problems in theoretical as well as applied mathematics [6]. The emergence of a large of software packages to perform computations in the algebra of the real quaternions [7], or more generally, Clifford algebra has been enhanced by the increasing interest in using quaternions and their applications in almost all applied sciences [8,9].

Definition 1. Let $0 < p < \infty$, $-2 < q < \infty$ and $0 < s < \infty$ and let f be an analytic function in $\mathbb{D}$. If

$$\|f\|^p_{F(p,q,s)} = \sup_{a\in\mathbb{D}} \int_{\mathbb{D}} |f'(z)|^p (1-|z|^2)^q g^s(z,a)\,dA(z) < \infty,$$

then $f \in F(p,q,s)$. Moreover, if

$$\lim_{|a|\to 1} \int_{\mathbb{D}} |f'(z)|^p (1-|z|^2)^q g^s(z,a)\,dA(z) = 0,$$

then $f \in F_0(p,q,s)$.

To introduce the meaning of hyperholomorphic functions, let $\mathbb{H}$ be the skew field of quaternions. The element $w \in \mathbb{H}$ can be written in the form:

$$w = w_0 + w_1 i + w_2 j + w_3 k, \quad w_0, w_1, w_2, w_3 \in \mathbb{R},$$

where $1, i, j, k$ are the basis elements of $\mathbb{H}$. For these elements we have the multiplication rules

$$i^2 = j^2 = k^2 = -1, ij = -ji = k, kj = -jk = i, ki = -ik = j.$$

The conjugate element $\bar{w}$ is given by $\bar{w} = w_0 - w_1 i - w_2 j - w_3 k$, and we have the property

$$w\bar{w} = \bar{w}w = \|w\|^2 = w_0^2 + w_1^2 + w_2^2 + w_3^2.$$

Moreover, we can identify each vector $\vec{x} = (x_0, x_1, x_2) \in \mathbb{R}^3$ with a quaternion x of the form

$$x = x_0 + x_1 i + x_2 j.$$

We will work in the unit ball in the real three-dimensional space, $\mathbb{B}_1(0) \subset \mathbb{R}^3$. We will consider functions f defined on $\mathbb{B}_1(0)$ with values in $\mathbb{H}$. We define a generalized Cauchy-Riemann operator D and it's conjugate $\overline{D}$ by

$$Df = \frac{\partial f}{\partial x_0} + i\frac{\partial f}{\partial x_1} + j\frac{\partial f}{\partial x_2},$$

and

$$\overline{D}f = \frac{\partial f}{\partial x_0} - i\frac{\partial f}{\partial x_1} - j\frac{\partial f}{\partial x_2}.$$

For these operators, we have

$$D\overline{D} = \overline{D}D = \Delta_3,$$

where Δ_3 is the Laplacian for functions defined over domains in $\mathbb{R}^3$. We denote by $\varphi_a(x) = (a-x)(1-\bar{a}x)^{-1}$, $|a| < 1$, the Möbius transform, which maps the unit ball onto itself.

Let

$$g(x,a) = \frac{1}{4\pi}\left(\frac{1}{|\varphi_a(x)|} - 1\right)$$

be the modified fundamental solution of the Laplacian in $\mathbb{R}^3$. Let $f : \mathbb{B} \mapsto \mathbb{H}$ be a hyperholomorphic function. Then [4]:

- $\mathcal{B}(f) = \sup_{x\in\mathbb{B}}(1-|x|^2)^{3/2}|\overline{D}f(x)|$,
- $Q_p(f) = \sup_{a\in\mathbb{B}} \int_{\mathbb{B}} |\overline{D}f(x)|^2 g^p(x,a)d\mathbb{B}_x$.

Definition 2. Let $0 < \alpha < \infty$. The hyperholomorphic α-Bloch space is defined as follows (see[2]):

$$\mathcal{B}^\alpha = \{f \in \ker D : \sup_{x\in\mathbb{B}}(1-|x|^2)^{\frac{3\alpha}{2}}|\overline{D}f(x)| < \infty\}.$$

The little α-Bloch type space $\mathcal{B}_0^\alpha$ is a subspace of $\mathcal{B}$ consisting of all $f \in \mathcal{B}^\alpha$ such that

$$\lim_{|x|\to 1^-}(1-|x|^2)^{\frac{3\alpha}{2}}|\overline{D}f(x)| = 0.$$

Definition 3. ([10]) Let f be quaternion-valued function in $\mathbb{B}$. For $0 < p < \infty$, $-2 < q < \infty$ and $0 < s < \infty$. If

$$\|f\|^p_{F(p,q,s)} = \sup_{a\in\mathbb{B}} \int_{\mathbb{B}} |\overline{D}f(x)|^p (1-|x|^2)^{\frac{3q}{2}} \left(1-|\varphi_a(x)|^2\right)^s d\mathbb{B}_x < \infty,$$

then $f \in F(p,q,s)$. Moreover, if

$$\lim_{|a|\to 1} \int_{\mathbb{B}} |\overline{D}f(x)|^p (1-|x|^2)^{\frac{3q}{2}} \left(1-|\varphi_a(x)|^2\right)^s d\mathbb{B}_x = 0,$$

then $f \in F_0(p,q,s)$.

The green function in $\mathbb{R}^3$ is defined as (see [11]):

$$G(x,a) = \frac{1-|\varphi_a(x)|^2}{|1-\bar{a}x|}.$$

We introduce following new definition of so called hyperholomorphic $F^\alpha_G(p,q,s)$ spaces.

Definition 4. Let $1 < \alpha,\ p < \infty$, $-2 < q < \infty$, and $s > 0$. Assume that f be hyperholomorphic function in the unit ball $\mathbb{B}_1(0)$. Then, $f \in F^\alpha_G(p,q,s)$, if

$$F_G^{\alpha}(p,q,s)=\left\{f\in kerD:\sup_{a\in\mathbb{B}_1(0)}\int_{\mathbb{B}_1(0)}|\overline{D}f(x)|^p(1-|x|^2)^{3\alpha q2+2s}\big(G(x,a)\big)^s d\mathbb{B}_x<\infty\right\}.$$

The space $F_{G,0}^{\alpha}(p,q,s)$ is subspace of $F_G^{\alpha}(p,q,s)$ consisting of all functions $f\in F_G^{\alpha}(p,q,s)$, such that

$$\lim_{|a|\to 1^-}\int_{\mathbb{B}_1(0)}|\overline{D}f(x)|^p(1-|x|^2)^{\frac{3\alpha q}{2}+2s}\big(G(x,a)\big)^s d\mathbb{B}_x=0.$$

Our objective in this article is twofold. First, we study the generalized quaternion space $F_G^{\alpha}(p,q,s)$ and characterize their relations to the quaternion $\mathcal{B}_0^{\alpha}$. Second, characterizations $F_G^{\alpha}(p,q,s)$ function space by the coefficients of Hadamard gap expansions. The following lemma, we will need in the sequel:

Lemma 5. *[12]. Let $0<R<1$, $1<q$, $a\in\mathbb{B}_1(0)$ and $f:\mathbb{B}_1(0)\longrightarrow\mathbb{H}$ be a hyperholomorphic function. Then*

$$|\overline{D}f(a)|^q\leq\frac{3\cdot 4^{2+q}}{\pi R^3(1-R^2)^{2q}(1-|a|^2)^3}\int_{\mathcal{M}(a,R)}\left|\overline{D}f(x)\right|^q d\mathbb{B}_x\,.$$

2. Power series expansions in $\mathbb{R}^3$

The major difference to power series in the complex case consists in the absence of regularity of the basic variable $x=x_0+x_1i+x_2j$ and of all of its natural powers x^n, $n=2,3,\ldots$. This means that we should expect other types of terms, which could be designated as generalized powers. We use a pair $\underline{y}=(y_1,y_2)$ of two regular variables given by

$$y_1=x_1-ix_0 \text{ and } y_2=x_2-jx_0,$$

and a multi-index $\nu=(\nu_1,\nu_2)$, $|\nu|=(\nu_1+\nu_2)$ to define the ν-power of $\underline{y}$ by a $|\nu|$-ary product [5,13,14].

Definition 6. Let ν_1 elements of the set $a_1,\ldots.,\ a_{|\nu|}$ be equal to y_1 and ν_2 elements be equal to y_2. Then the ν-power of $\underline{y}$ is defined by

$$\underline{y}:=\frac{1}{|\nu|!}\sum_{(i_1,\ldots,i_{|\nu|})\in\pi(1,\ldots|\nu|)}a_{i1}a_{i2}\ldots a_{i_{|\nu|}},\tag{1}$$

where the sum runs over all permutations of $(1,\ldots.,|\nu|)$.

The general form of the Taylor series of left monogenic functions in the neighborhood of the origin is given as [14]:

$$P(\underline{y}):=\sum_{n=0}^{\infty}\Big(\sum_{|\nu|=n}\underline{y}^{\nu}c_{\nu}\Big),\quad c_{\nu}\in\mathbb{H}.\tag{2}$$

Theorem 7. *[5,15]) Let $g(x)$ be left hyperholomorphic with the Taylor series (2). Then*

$$\left|\frac{1}{2}\overline{D}g(x)\right|\leq\sum_{n=1}^{\infty}n\Big(\sum_{|\nu|=n}|c_{\nu}|\Big)|x|^{n-1}.\tag{3}$$

We introduce the notation $\mathbf{H}_n(x):=\sum_{|\nu|=n}\underline{y}^{|\nu|}c_{\nu}$ and consider monogenic functions composed by $\mathbf{H}_n(x)$ in the following form:

$$f(x)=\sum_{n=0}^{\infty}\mathbf{H}_n(x)b_n,\qquad b_n\in\mathbb{H}.$$

Using (3), we have

$$\left|\frac{1}{2}\overline{D}f(x)\right|\leq\sum_{n=1}^{\infty}n\Big(\sum_{|\nu|=n}|c_{\nu}|\Big)|b_n||x|^{n-1}.\tag{4}$$

This is the motivation for another notation,

$$a_n := \left(\sum_{|\nu|=n} |c_\nu| \right) |b_n| \quad (a_n \geq 0), \tag{5}$$

finally, we have

$$\left| \frac{1}{2} \overline{D} f(x) \right| \leq \sum_{n=1}^{\infty} n a_n |x|^{n-1}. \tag{6}$$

3. Lacunary series expansions in $F_G^\alpha(p,q,s)$ spaces

In this section, we give a sufficient and necessary condition for the hyperholomorphic function f on $\mathbb{B}_1(0)$ of $\mathbb{R}^3$ with Hadamard gaps to belong to the weighted hyperholomorphic $F_G^\alpha(p,q,s)$ spaces. The function

$$f(r) = \sum_{k}^{\infty} a_k r^{n_k} \qquad (\, n_k \in \mathbb{N};\ \forall\, k \in \mathbb{N}) \tag{7}$$

belong to the Hadamard gap class (Lacunary series) if there exists a constant $\lambda > 1$ such that $\frac{n_{k+1}}{n_k} \geq \lambda,\ \forall\, k \in \mathbb{N}$. Characterizations in higher dimensions using several complex variables and quaternion sense [16–18].

Theorem 8. *Let $f(r) = \sum_{n=1}^{\infty} a_n r^n$, with $a_n \geq 0$. If $\alpha > 0,\ p > 0$. Then*

$$\int_0^1 (1-r)^{\alpha-1} (f(r))^p \, dr \approx \sum_{n=0}^{\infty} 2^{-n\alpha}\ t_n^p, \tag{8}$$

where $t_n = \sum_{k \in I_n} a_k,\ n \in \mathbb{N},\ I_n = \{k : 2^n \leq k < 2^{n+1};\ \ k \in \mathbb{N}\}$.

Proof. The prove of this theorem can be obtained easily from Theorem 2.5 of [19] with the same steps. □

Theorem 9. *Let $\alpha,\ p \geq 1,\ -2 < q < \infty,\ s > 0$, and $I_n = \{k : 2^n \leq k < 2^{n+1}; k \in \mathbb{N}\}$. Suppose that $f(x) = \sum_{n=0}^{\infty} H_n(x) b_n, b_n \in \mathbb{H}$, where $H_n(x)$ be homogenous hyperholomorphic polynomials of degree n, and let a_n be define as before in (5). If*

$$\sum_{n=0}^{\infty} 2^{-n(\frac{3}{2}\alpha q + s - p + 1)} \Big(\sum_{k \in I_n} |a_k| \Big)^p < \infty, \tag{9}$$

then

$$\sup_{a \in \mathbb{B}_1(0)} \int_{\mathbb{B}_1(0)} \left| \frac{1}{2} \overline{D} f(x) \right|^p (1-|x|^2)^{\frac{3\alpha q}{2} + 2s} \big(G(x,a) \big)^s d\mathbb{B}_x < \infty, \tag{10}$$

and $f \in F_G^\alpha(p,q,s)$.

Proof. Suppose that (9) holds. Using the equality

$$G(x,a) = \frac{1 - |\varphi_a(x)|^2}{|1 - \bar{a}x|} = \frac{(1-|a|^2)(1-|x|^2)}{|1-\bar{a}x|^3}, \tag{11}$$

where

$$1 - |x| \leq |1 - \bar{a}x| \leq 1 + |x|, \qquad 1 - |a| \leq |1 - \bar{a}x| \leq 1 + |a| \leq 2. \tag{12}$$

Then, we get

$$\begin{aligned}
&\int_{\mathbb{B}_1(0)}\left|\frac{1}{2}\overline{D}f(x)\right|^p(1-|x|^2)^{\frac{3\alpha q}{2}+2s}(G(x,a))^s d\mathbb{B}_x\\
&=\int_{\mathbb{B}_1(0)}\left|\frac{1}{2}\overline{D}\left(\sum_{n=0}^{\infty}H_n(x)b_n\right)\right|^p(1-|x|^2)^{\frac{3\alpha q}{2}+2s}\frac{(1-|a|^2)^s(1-|x|^2)^s}{|1-\overline{a}x|^{3s}}d\mathbb{B}_x\\
&\leq\int_{\mathbb{B}_1(0)}\left(\sum_{n=0}^{\infty}na_nx^{n-1}\right)^p(1-|x|^2)^{\frac{3\alpha q}{2}+2s}\frac{(1-|a|^2)^s(1-|x|^2)^s}{(1-|a|)^s(1-|x|)^{2s}}d\mathbb{B}_x\\
&\leq 2^{\frac{3\alpha q}{2}+4s}\int_0^1\left(\sum_{n=0}^{\infty}na_nr^{n-1}\right)^p(1-r)^{3\alpha q2+s}r^2dr\\
&\leq\lambda\int_0^1\left(\sum_{n=0}^{\infty}na_nr^{n-1}\right)^p(1-r)^{\frac{3\alpha q}{2}+s}dr. \qquad (13)
\end{aligned}$$

Using Theorem 8 in (13), we deduced that

$$\begin{aligned}
\int_{\mathbb{B}_1(0)}\left|\frac{1}{2}\overline{D}f(x)\right|^p(1-|x|^2)^{\frac{3\alpha q}{2}+2s}(G(x,a))^s d\mathbb{B}_x &\leq \lambda\int_0^1\left(\sum_{n=0}^{\infty}na_nr^{n-1}\right)^p(1-r)^{\frac{3\alpha q}{2}+s}dr\\
&\leq \lambda\sum_{n=0}^{\infty}2^{-n(3\alpha q2+s+1)}t_n^P. \qquad (14)
\end{aligned}$$

Since

$$t_n=\sum_{k\in I_n}ka_k<2^{n+1}\sum_{k\in I_n}a_k,$$

we obtain that,

$$\int_{\mathbb{B}_1(0)}\left|\frac{1}{2}\overline{D}f(x)\right|^p(1-|x|^2)^{\frac{3\alpha q}{2}+2s}(G(x,a))^s d\mathbb{B}_x \leq \lambda_1\sum_{n=0}^{\infty}2^{-n(\frac{3\alpha q}{2}+s-p+1)}\Big(\sum_{k\in I_n}|a_k|\Big)^p.$$

Therefore, we have

$$\|f\|_{F_G^{\alpha}(p,q,s)}\leq\lambda_1\sum_{n=0}^{\infty}2^{-n(\frac{3\alpha q}{2}+s-p+1)}\Big(\sum_{k\in I_n}|a_k|\Big)^p<\infty,$$

where λ_1 is a constant. Then, the last inequality implies that $f\in F_G^{\alpha}(p,q,s)$ and the proof of our theorem is completed. □

For the converse of Theorem 9, we consider the following theorem.

Proposition 10. *(see [5]) Let $\alpha=(\alpha_1,\alpha_2),\alpha_i\in\mathbb{R},\ i=1,2$ be the vector of real coefficients defining $\mathrm{H}_{n,\alpha}(x)=(y_1\alpha_1+y_2\alpha_2)^n$. Suppose that $|\alpha|^2=\alpha_1^2+\alpha_2^2\neq 0$. Then,*

$$\|\mathrm{H}_{n,\alpha}\|^p_{L_p(\partial\mathbb{B}_1)}=2\pi\sqrt{\pi}|\alpha|^{np}\frac{\Gamma(\frac{n}{2}p+1)}{\Gamma(\frac{n}{2}p+\frac{3}{2})},\quad \text{where } 0<p<\infty. \qquad (15)$$

Moreover, we have (see [5])

$$\frac{\|-\frac{1}{2}\overline{D}\mathrm{H}_{n,\alpha}\|^p_{L_p(\partial\mathbb{B}_1)}}{\|\mathrm{H}_{n,\alpha}\|^p_{L_p(\partial\mathbb{B}_1)}}=n^p\frac{\mathbf{B}\left(\frac{1}{2},\frac{n-1}{2}p+1\right)}{\mathbf{B}\left(\frac{1}{2},\frac{n}{2}p+1\right)}\geq\lambda n^p>0, \qquad (16)$$

where, $\mathbf{B}\left(\frac{1}{2},\frac{n-1}{2}p+1\right)=\frac{\Gamma(\frac{1}{2})\Gamma(\frac{n-1}{2}p+1)}{\Gamma(\frac{n-1}{2}p+\frac{3}{2})}$, *and* $\lim_{n\to\infty}\frac{\mathbf{B}(\frac{1}{2},\frac{n-1}{2}p+1)}{\mathbf{B}(\frac{1}{2},\frac{n}{2}p+1)}=1.$

Corollary 11. *[5] Assume that $p \geq 2$. Then,*

$$\frac{\| -\frac{1}{2}\overline{D}\mathrm{H}_{n,\alpha}\|^2_{L_2(\partial\mathbb{B}_1)}}{\|\mathrm{H}_{n,\alpha}\|^2_{L_p(\partial\mathbb{B}_1)}} \geq \lambda n^{\frac{2+3p}{2p}}. \tag{17}$$

Theorem 12. *Let $\alpha \geq 1, 2 \leq p < \infty, -2 < q < \infty, s > 0$, and $0 < |x| = r < 1$. If*

$$f(x) = \left(\sum_{n=0}^{\infty} \frac{\mathrm{H}_{n,\alpha}}{(1-|x|^2)^{\frac{8s+p}{4p}} \|\mathrm{H}_{n,\alpha}\|_{L_p(\partial\mathbb{B}_1)}} a_n \right) \in F_G^{\alpha}(p,q,s). \tag{18}$$

Then,

$$\sum_{n=0}^{\infty} 2^{-n(\frac{3}{2}\alpha q+s-p+1)} \Big(\sum_{k\in I_n} |a_k|\Big)^p < \infty. \tag{19}$$

Proof. Since

$$\begin{aligned}
\|f\|_{F_G^{\alpha}(p,q,s)} &= \sup_{a\in\mathbb{B}_1(0)} \int_{\mathbb{B}_1(0)} |\overline{D}f(x)|^p (1-|x|^2)^{\frac{3\alpha q}{2}+2s} (G(x,a))^s d\mathbb{B}_x \\
&= \sup_{a\in\mathbb{B}_1(0)} \int_{\mathbb{B}_1(0)} |\overline{D}f(x)|^p (1-|x|^2)^{\frac{3\alpha q}{2}+2s} \left(\frac{(1-|x|^2)(1-|a|^2)}{|1-\bar{a}x|^3} \right)^s d\mathbb{B}_x \\
&\geq \sup_{a\in\mathbb{B}_1(0)} \int_{\mathbb{B}_1(0)} |\overline{D}f(x)|^p (1-|x|^2)^{\frac{3\alpha q}{2}+3s} d\mathbb{B}_x \quad (\text{where } a=0).
\end{aligned} \tag{20}$$

Hence, we have

$$\begin{aligned}
\|f\|_{F_G^{\alpha}(p,q,s)} &\geq \int_{\mathbb{B}_1(0)} |-\frac{1}{2}\overline{D}f(x)|^p (1-|x|^2)^{\frac{3\alpha q}{2}+3s} d\mathbb{B}_x \quad (\text{where } a=0). \\
&= \int_{\mathbb{B}_1(0)} \left| \sum_{n=0}^{\infty} \left[\frac{-\frac{1}{2}\overline{D}\mathrm{H}_{n,\alpha}}{(1-|x|^2)^{\frac{8s+p}{4p}} \|\mathrm{H}_{n,\alpha}\|_{L_p(\partial\mathbb{B}_1)}} \right] a_n \right|^p (1-|x|^2)^{\frac{3\alpha q}{2}+3s} d\mathbb{B}_x.
\end{aligned} \tag{21}$$

where $\left[\frac{-\frac{1}{2}\overline{D}\mathrm{H}_{n,\alpha}}{\|\mathrm{H}_{n,\alpha}\|_{L_p(\partial\mathbb{B}_1)}}\right]$ is a homogeneous hyperholomorphic polynomial of degree n-1 and it can be written in the form

$$\left[\frac{-\frac{1}{2}\overline{D}\mathrm{H}_{n,\alpha}}{\|\mathrm{H}_{n,\alpha}\|_{L_p(\partial\mathbb{B}_1)}}\right] = r^{(n-1)} \Phi_n(\phi_1,\phi_2), \tag{22}$$

where

$$\Phi_n(\phi_1,\phi_2) := \left(\left[\frac{-\frac{1}{2}\overline{D}\mathrm{H}_{n,\alpha}}{\|\mathrm{H}_{n,\alpha}\|_{L_p(\partial\mathbb{B}_1)}}\right] \right)_{\partial\mathbb{B}_1}. \tag{23}$$

Now, by the inner product $\langle f,g\rangle_{\partial\mathbb{B}_1(0)} = \int_{\partial\mathbb{B}_1(0)} \overline{f(x)} g(x) d\Gamma_x$, the orthogonality of the spherical monogenic $\Phi_n(\phi_1,\phi_2)$ (see [20]) in $L_2(\partial\mathbb{B}_1(0))$. From (22) and (23) to (21), we have

$$\begin{aligned}
&\int_{\mathbb{B}_1(0)} \left| \sum_{n=0}^{\infty} \left[\frac{-\frac{1}{2}\overline{D}\mathrm{H}_{n,\alpha}}{(1-|x|^2)^{\frac{8s+p}{4p}} \|\mathrm{H}_{n,\alpha}\|_{L_p(\partial\mathbb{B}_1)}} \right] a_n \right|^p (1-|x|^2)^{\frac{3\alpha q}{2}+3s} d\mathbb{B}_x \\
&= \int_0^1 \int_{\partial\mathbb{B}_1(0)} \left(\left| \sum_{n=0}^{\infty} \frac{r^{n-1}}{(1-r^2)^{\frac{8s+p}{4p}}} \Phi_n(\phi_1,\phi_2) a_n \right|^2 \right)^{\frac{p}{2}} r^2 (1-r^2)^{\frac{3\alpha q}{2}+3s} d\Gamma_x dr \\
&= \int_0^1 \int_{\partial\mathbb{B}_1(0)} \left(\sum_{n=0}^{\infty} \sum_{j=0}^{\infty} \overline{a_n} \frac{r^{2n-2}}{(1-r^2)^{\frac{8s+p}{2p}}} \overline{\Phi_n}(\phi_1,\phi_2) \Phi_j(\phi_1,\phi_2) a_j \right)^{\frac{p}{2}} r^2 (1-r^2)^{\frac{3\alpha q}{2}+3s} d\Gamma_x dr = \mathrm{L}.
\end{aligned} \tag{24}$$

From Hölder's inequality, we have

$$\int_{\partial\mathbb{B}_1(0)} |f(x)|^p d\Gamma_x \geq (4\pi)^{1-p} \left| \int_{\partial\mathbb{B}_1(0)} f(x) d\Gamma_x \right|^p, \quad (where\ 1 \leq p < \infty). \tag{25}$$

From (25), for $2 \leq p < \infty$, we have

$$\begin{aligned} \mathrm{L} &\geq (4\pi)^{1-\frac{p}{2}} \int_0^1 \left(\sum_{n=0}^{\infty} |a_n|^2 \frac{r^{2n-2}}{(1-r^2)^{\frac{8s+p}{2p}}} \|\Phi_n(\phi_1,\phi_2)\|^2_{L_2(\partial\mathbb{B}_1)} \right)^{\frac{p}{2}} r^2(1-r^2)^{\frac{3\alpha q}{2}+3s} dr \\ &\geq (4\pi)^{1-\frac{p}{2}} \int_0^1 \left(\sum_{n=0}^{\infty} |a_n|^2 r^{2n-2} \|\Phi_n(\phi_1,\phi_2)\|^2_{L_2(\partial\mathbb{B}_1)} \right)^{\frac{p}{2}} r^3(1-r^2)^{\frac{3\alpha q}{2}+s-\frac{p}{4}} dr \end{aligned} \tag{26}$$

From Corollary 11, we have

$$\|\Phi_n(\phi_1,\phi_2)\|^2_{L_2(\partial\mathbb{B}_1)} = \frac{\|-\frac{1}{2}\overline{D}\mathrm{H}_{n,\alpha}\|_{L_2(\partial\mathbb{B}_1)}}{\|\mathrm{H}_{n,\alpha}\|_{L_p(\partial\mathbb{B}_1)}} \geq \lambda n^{\frac{2+3p}{2p}} \geq \lambda n^{\frac{3}{2}}.$$

Then, from above we have

$$\begin{aligned} \mathrm{L} &\geq (4\pi)^{1-\frac{p}{2}} \lambda \int_0^1 \left(\sum_{n=0}^{\infty} n^{\frac{3}{2}} |a_n|^2 r^{2n-2} \right)^{\frac{p}{2}} r^3(1-r^2)^{\frac{3\alpha q}{2}+s-\frac{p}{4}} dr \\ &= \lambda_1 \int_0^1 \left(\sum_{n=0}^{\infty} n^{\frac{3}{2}} |a_n|^2 r^{2n-2} \right)^{\frac{p}{2}} r^3(1-r^2)^{\frac{3\alpha q}{2}+s-\frac{p}{4}} dr \\ &= \frac{\lambda_1}{2} \int_0^1 \left(\sum_{n=0}^{\infty} n^{\frac{3}{2}} |a_n|^2 \xi^{n-1} \right)^{\frac{p}{2}} \xi(1-\xi)^{\frac{3\alpha q}{2}+s-\frac{p}{4}} d\xi \\ &\geq \lambda_3 \int_0^1 \left(\sum_{n=0}^{\infty} n^{\frac{3}{2}} |a_n|^2 \xi^{n-1} \right)^{\frac{p}{2}} (1-\xi)^{\frac{3\alpha q}{2}+s-\frac{p}{4}} d\xi, \end{aligned} \tag{27}$$

where λ_j, $j = 1,2,3$, are constants do not depending on n.

Now, by applying Theorem 8 in equation (27), we deduced that

$$\|f\|_{F_G^\alpha(p,q,s)} \geq \mathrm{L} \geq \frac{\lambda_3}{k} \sum_{n=0}^{\infty} 2^{-n(\frac{3\alpha q}{2}+s-\frac{p}{4}+1)} \Big(\sum_{k\in I_n} k^{\frac{3}{2}} |a_k|^2 \Big)^{\frac{p}{2}}, \tag{28}$$

where

$$\sum_{k\in I_n} k^{\frac{3}{2}} |a_k|^2 > \left(2^n\right)^{\frac{3}{2}} \left(\sum_{k\in I_n} |a_k|^2 \right)^{\frac{p}{2}}.$$

Then,

$$\|f\|_{F_G^\alpha(p,q,s)} \geq \mathrm{L} \geq C \sum_{n=0}^{\infty} 2^{-n(\frac{3\alpha q}{2}+s-p+1)} \Big(\sum_{k\in I_n} |a_k|^2 \Big)^{\frac{p}{2}}, \tag{29}$$

From [21], we have

$$\sum_{n=0}^{N} a_n^p \leq \left(\sum_{n=0}^{N} a_n^p \right)^p \leq N^{p-1} \sum_{n=0}^{N} a_n^p.$$

Then, we have

$$\|f\|_{F_G^\alpha(p,q,s)} \geq \mathrm{L} \geq C_1 \sum_{n=0}^{\infty} 2^{-n(\frac{3\alpha q}{2}+s-p+1)} \Big(\sum_{k\in I_n} |a_k| \Big)^{p}, \tag{30}$$

where C_1 be a constants which do not depend on n. Then,

$$\sum_{n=0}^{\infty} 2^{-n(\frac{3\alpha q}{2}+s-p+1)} \left(\sum_{k\in I_n} |a_k| \right)^{p} < \infty. \tag{31}$$

This completes the proof of theorem. □

Theorem 13. *Let $\alpha \geq 1, 2 \leq p < \infty, -2 < q < \infty$, and $s > 0$, then we have*

$$f(x) = \left(\sum_{n=0}^{\infty} \frac{\mathrm{H}_{n,\alpha}}{(1-|x|^2)^{\frac{8s+p}{4p}} \|\mathrm{H}_{n,\alpha}\|_{L_p(\partial\mathbb{B}_1)}} a_n \right) \in F^{\alpha}_{G}(p,q,s), \tag{32}$$

if and only if,

$$\sum_{n=0}^{\infty} 2^{-n(\frac{3}{2}\alpha q+s-p+1)} \Big(\sum_{k \in I_n} |a_k| \Big)^p < \infty. \tag{33}$$

Proof. This theorem can be proved directly from Theorem 9 and Theorem 12. □

4. Conclusion

We have introduce a new class of hyperholomorphic functions, which is also called $F^{\alpha}_{G}(p,q,s)$ spaces. For this class, we give some characterizations of the hyperholomorphic $F^{\alpha}_{G}(p,q,s)$ functions by the coefficients of certain lacunary series expansions in quaternion analysis.

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