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Analysis of the small oscillations of a heavy barotropic gas filling an elastic body with negligible density

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Abstract: In this work, we study the small oscillations of a system formed by an elastic container with negligible density and a heavy barotropic gas (or a compressible fluid) filling the container. By means of an auxiliary problem, that requires a careful mathematical study, we deduce the problem to a problem for a gas only. From its variational formulation, we prove that is a classical vibration problem.

Keywords: Barotropic gas, small oscillations, mixed boundary conditions, vibration problem, variational and spectral methods.

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1. Introduction

The problem of the small oscillations of a heavy homogeneous inviscid liquid in an open rigid container has been the subject, from the pioneering work by Moiseyev [1], of numerous papers, that are analyzed in the books [2–4].

The same problem in the case of an elastic container is studied in the book [5]. Recently, we have solved the problem of the small oscillations of an heterogeneous liquid in an elastic container [6].

In this work, we study the problem of the small oscillations of a system formed by a heavy barotropic gas (or a compressible fluid) and an elastic body with negligible density, circumitance that can happen in the transport of fluids. At first, we establish the equations of motion of the system body-gas and the boundaries conditions. Afterwards, introducing an auxiliary problem, that requires a careful mathematical discussion, and that is the problem of the motion of the body when the motion of the gas is known, we show a linear operator depending on the elasticity of the body, that permits us to reduce the problem to a problem for the gas only. From the variational equation of this last problem, we prove that it is a classical vibration problem.

2. Position of the problem

We consider, in the field of the gravity, an elastic body with negligible density, that occupies in the equilibrium position a domain Ω' bounded by a fixed external surface S and an internal surface Σ. The interior Ω of this surface is completely filled by a heavy barotropic gas.

We choose orthogonal axes $Ox_1x_2x_3$, Ox_3 vertical directed upwards and we denote by $\overrightarrow{n}$ the unit vector normal to the surfaces. We are going to study the small oscillations of the system elastic body-gas about its equilibrium position, in the framework of the linear theory.

3. The equations of the problem

3.1. The equations of the elastic body with negligible density

Let $\vec{u}'(x_i)$ the (small) displacement of the particle of the body from the natural state to the equilibrium position. The equilibrium equations are:

$$0 = \frac{\partial \sigma'_{ij}(\vec{u}')}{\partial x_j} \qquad \text{in} \qquad \Omega' \ \ (i,j = 1,2,3) \tag{1}$$

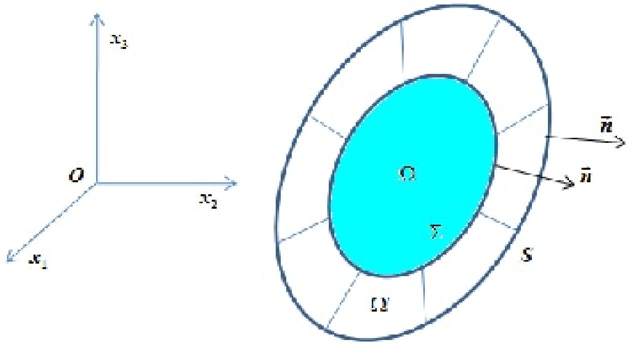

Figure 1. Model of the system

and the boundary conditions are

$$\vec{u}'_{|S} = 0 \qquad ; \qquad \sigma'_{ij}(\vec{\hat{u}}')n_j = -p_0 n_i \qquad \text{on} \quad \Sigma\,, \tag{2}$$

where p_0 is the pressure of the gas in the equilibrium position and where we have set:

$$\sigma'_{ij}(\vec{\hat{u}}') = \lambda'\delta_{ij}\text{div}\vec{\hat{u}}' + 2\mu'\epsilon'_{ij}(\vec{\hat{u}}') \qquad ; \qquad \epsilon'_{ij}(\vec{\hat{u}}') = \frac{1}{2}\left(\frac{\partial\hat{u}'_i}{\partial x_j} + \frac{\partial\hat{u}'_j}{\partial x_i}\right)$$

λ' and μ' are the Lame's coefficients; $\sigma'_{ij}(\vec{\hat{u}}')$ and $\epsilon'_{ij}(\vec{\hat{u}}')$ are the components of the stress tensor and the strain tensor respectively.

Now, let $\vec{u}'(x_i, t)$ the displacement of a particle from its equilibrium position to its position at the instant t. We have

$$0 = \frac{\partial\sigma'_{ij}(\vec{\hat{u}}' + \vec{u}')}{\partial x_j} \qquad \text{in} \qquad \Omega'$$

and consequently

$$0 = \frac{\partial\sigma'_{ij}(\vec{u}')}{\partial x_j} \qquad \text{in} \qquad \Omega'\,, \tag{3}$$

and in the same manner

$$\vec{u}'_{|S} = 0\,. \tag{4}$$

Let $\vec{u}\,(x_i, t)$ the displacement of a particle of the gas from its equilibrium position to its position at the instant t; we must have the kinematic condition:

$$u'_{n|\Sigma} = u_{n|\Sigma}\,, \tag{5}$$

where we have set $u_n = \vec{u}\cdot\vec{n}$.

3.2. The equations of the barotropic gas

Let ρ^*, P the density and the pressure of the gas that are related by

$$P = \mathscr{P}\left(\rho^*\right)\,, \tag{6}$$

where $\mathscr{P}$ is a given smooth increasing function. If ρ_0 is the density in the equilibrium postion, we have

$$p_0 = \mathscr{P}\left(\rho_0\right)$$

and the equilibrium equation

$$\overrightarrow{\text{grad}}p_0 = -\rho_0 g\vec{x}_3 \tag{7}$$

Then, p_0 and ρ_0 are functions of x_3 only and we have

$$\frac{\mathrm{d}p_0(x_3)}{\mathrm{d}x_3} = -\rho_0(x_3)g\,. \tag{8}$$

Setting classically
$$c_0^2\left(x_3\right)=\mathscr{P}'\left(\rho_0\left(x_3\right)\right)\ , \tag{9}$$
we obtain
$$c_0^2\left(x_3\right)\rho_0'\left(x_3\right)=-\rho_0\left(x_3\right)g\ . \tag{10}$$

It is a differential equation of the first order that must be verified by $\rho_0\left(x_3\right)$. The equation of the motion of the gas are, besides (6):

$$\rho^*\ddot{\vec{u}}=-\overrightarrow{\text{grad}}P-\rho^* g\vec{x}_3 \quad \text{(Euler's equation)} \quad \text{in} \quad \Omega\,, \tag{11}$$
$$\frac{\partial\rho^*}{\partial t}+\text{div}(\rho^*\dot{\vec{u}})=0 \quad \text{(continuity equation)} \quad \text{in} \quad \Omega\,. \tag{12}$$

Since, we study the small motions of the gas about its equilibrium position, we set

$$\rho^*=\rho_0(x_3)+\tilde{\rho}(x_i,t)+\cdots\ ,$$
$$P=p_0(x_3)+p(x_i,t)+\cdots\ .$$

The $\tilde{\rho}$ and the dynamic pressure p are of the first order with respect to the amplitude of the oscillations, the dots represent terms of order greater than one. We have, at the first order

$$\frac{\partial\tilde{\rho}}{\partial t}+\text{div}(\rho_0(x_3)\dot{\vec{u}})=0\ ;$$

integrating between the datum of the equilibrium position and the instant t, we have

$$\tilde{\rho}=-\text{div}\left[\rho_0(x_3)\vec{u}\right]\ . \tag{13}$$

Using (6), we have
$$p_0(x_3)+p+\cdots=\mathscr{P}\left(\rho_0(x_3)+\tilde{\rho}+\cdots\right)$$
and then
$$p=-c_0^2(x_3)\text{div}\left[\rho_0(x_3)\vec{u}\right]\ . \tag{14}$$

The Euler's Equation can be written

$$\begin{aligned}\rho_0\ddot{\vec{u}}+\cdots &= -\overrightarrow{\text{grad}}\left(p_0+p+\cdots\right)-\left(\rho_0-\text{div}\left(\rho_0\vec{u}\right)+\cdots\right)g\vec{x}_3\\ &= \overrightarrow{\text{grad}}\left(c_0^2\text{div}\left(\rho_0\vec{u}\right)\right)+g\text{div}\left(\rho_0\vec{u}\right)\vec{x}_3+\cdots\ ,\end{aligned}$$

and, using the equation (10), finally we get

$$\ddot{\vec{u}}=\overrightarrow{\text{grad}}\left(\frac{c_0^2(x_3)}{\rho_0(x_3)}\text{div}\left(\rho_0(x_3)\vec{u}\right)\right)\ , \tag{15}$$

which is the equation that contains $\vec{u}$ only.

3.3. The dynamic conditions on the surface Σ_t

Let M a point of Σ. We denote by M_g, M_s the particles of the gas and of the elastic body that are in M at the instant $t=0$. These particles come in M_g', M_s' on Σ_t at the instant t:

$$\overrightarrow{MM_g'}=\vec{u} \qquad ; \qquad \overrightarrow{MM_s'}=\vec{u}'$$

In linear theory, we admit that the unit vectors normal to Σ_t in M_g' and M_s' are equipollent to the unit vector $\vec{n}$ normal in M to Σ and that the pressure of the gas P in M_g' is equal to the pressure of the gas in M', intersection of Σ_t with the normal in M to Σ.

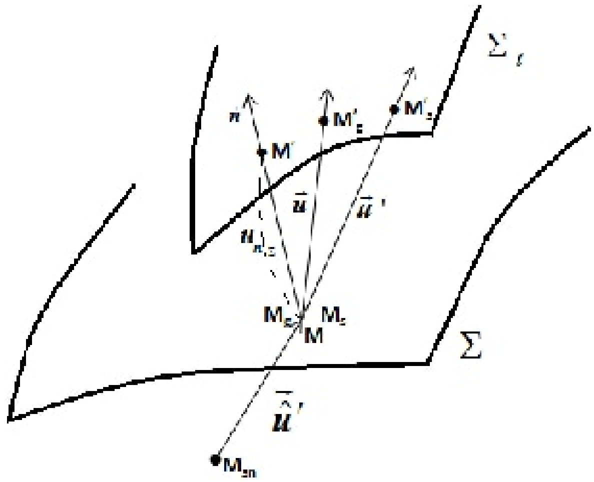

Figure 2. Configuations of Σ and Σ_t

The dynamic conditions on Σ_t are

$$\sigma'_{ij}(\hat{\vec{u}}' + \vec{u}')n_j = -P(M',t)n_i \,.$$

Or, using the second condition (2):

$$\sigma'_{ij}(\vec{u}')n_j = -[P(M',t) - p_0(M)] \cdot n_i \qquad \text{on } \Sigma \,.$$

We have

$$P(M',t) = P(M + u_{n|\Sigma}\vec{n},t) = \mathcal{P}(M,t) + \overrightarrow{\text{grad}} P(M) \cdot u_{n|\Sigma}\vec{n} + \cdots$$

Since $u_{n|\Sigma}$ is of the first order, we can, in linear theory, replace $\overrightarrow{\text{grad}} P(M,t)$ by

$$\overrightarrow{\text{grad}} p_0 = -\rho_0 \, g\vec{x}_3 \,,$$

so that

$$P(M',t) = P(M,t) - \rho_0 g \, u_{n|\Sigma} \, n_{3|\Sigma} + \cdots$$

and finally

$$\sigma'_{ij}(\vec{u}')n_j = [-p(M,t) + \rho_{0|\Sigma} g n_{3|\Sigma} \, u_{n|\Sigma}]n_i \qquad \text{on } \Sigma \,. \tag{16}$$

Let us call $\overrightarrow{T}_t \, (\vec{u}')_{|\Sigma}$ the tangential stress and $T_n \, (\vec{u}')_{|\Sigma}$ the normal stress; we have

$$\overrightarrow{T}_t \, (\vec{u}')_{|\Sigma} = 0 \quad ; \quad T_n \, (\vec{u}')_{|\Sigma} = -p_{|\Sigma} + \rho_{0|\Sigma} g n_{3|\Sigma} \, u_{n|\Sigma} \,. \tag{17}$$

4. The auxiliary problem

Step 1.

We introduce the following auxiliary problem:

$$-\frac{\partial \sigma'_{ij}(\vec{u}')}{\partial x_j} = 0 \quad \text{in} \quad \Omega' \quad ; \quad \vec{u}'_{|S} = 0 \quad ; \quad u'_{n|\Sigma} = u_{n|\Sigma} \quad ; \quad \overrightarrow{T}_t \, (\vec{u}')_{|\Sigma} = 0 \,, \tag{18}$$

where $u_{n|\Sigma}$ is considered as a datum. It is the problem of the motion of an elastic body when the motion of the gas is known and we seek the solution of this auxiliary problem in the space.

$$\widehat{\Xi}^1(\Omega') \overset{\text{def}}{=} \left\{ \vec{u}' \in \Xi^1(\Omega') \overset{\text{def}}{=} \left[H^1(\Omega') \right]^3 ; \; \vec{u}'_{|S} = 0 \right\} \,.$$

Then $u'_{n|\Sigma} \in H^{1/2}(\Sigma)$ and consequently, we suppose that $u_{n|\Sigma} \in H^{1/2}(\Sigma)$.

Step 2.

Let $\overrightarrow{\Phi}$ an element of $\widehat{\Xi}^1(\Omega')$ such that $\Phi_{n|\Sigma} = u_{n|\Sigma} \in H^{1/2}(\Sigma)$.

In the following, we will see the construction of such $\overrightarrow{\Phi}$. We denote by V_0 the subspace of $\hat{\Xi}^1(\Omega')$ defined by

$$V_0 = \left\{ \vec{v}_0 \in \widehat{\Xi}^1(\Omega') \quad ; \quad v_{0n|\Sigma} = 0 \right\}$$

and we seek the solution $\vec{u}'$ of the auxiliary problem in the form

$$\vec{u}' = \overrightarrow{\Phi} + \vec{v}_0 .$$

The auxiliary problem (18) becomes a problem for $\vec{u}_0 \in V_0$:

$$-\frac{\partial \sigma'_{ij}(\vec{u}_0)}{\partial x_j} = \frac{\partial \sigma'_{ij}(\overrightarrow{\Phi})}{\partial x_j} \quad \text{in} \quad \Omega' \quad ; \quad u_{0n|\Sigma} = 0 \quad ; \quad \overrightarrow{T}_t\,(\vec{u}'_0)_{|\Sigma} = -\overrightarrow{T}_t(\overrightarrow{\Phi})_{|\Sigma} . \tag{19}$$

Let us seek a variational formulation of this problem. We have, for each $\vec{v}_0 \in V_0$:

$$-\int_{\Omega'} \frac{\partial \sigma'_{ij}(\vec{u}_0)}{\partial x_j} \cdot \bar{v}_{0i}\, \mathrm{d}\Omega' = \int_{\Omega'} \frac{\partial \sigma'_{ij}(\overrightarrow{\Phi})}{\partial x_j} \cdot \bar{v}_{0i}\, \mathrm{d}\Omega'$$

or

$$-\int_{\Omega'} \left[\frac{\partial}{\partial x_j}[\sigma'_{ij}(\vec{u}_0)\bar{v}_{0i}] - \sigma'_{ij}(\vec{u}_0)\epsilon'_{ij}(\bar{\vec{v}}_0) \right] \mathrm{d}\Omega'$$
$$= \int_{\Omega'} \left[\frac{\partial}{\partial x_j}[\sigma'_{ij}(\overrightarrow{\Phi})\bar{v}_{0i}] - \sigma'_{ij}(\overrightarrow{\Phi})\epsilon'_{ij}(\bar{\vec{v}}_0) \right] \mathrm{d}\Omega' ,$$

or, using the Green's formula and denoting by $\vec{n}_e$, the external normal unit vector to $\partial\Omega'$:

$$-\int_S \sigma'_{ij}(\vec{u}_0) n_{ej} \bar{v}_{0i}\, \mathrm{d}S - \int_\Sigma \sigma'_{ij}(\vec{u}_0) n_{ej} \bar{v}_{0i}\, \mathrm{d}\Sigma + \int_{\Omega'} \sigma'_{ij}(\vec{u}_0)\epsilon'_{ij}(\bar{\vec{v}}_0)\, \mathrm{d}\Omega'$$
$$= \int_S \sigma'_{ij}(\overrightarrow{\Phi}) n_{ej} \bar{v}_{0i}\, \mathrm{d}S + \int_\Sigma \sigma'_{ij}(\overrightarrow{\Phi}) n_{ej} \bar{v}_{0i}\, \mathrm{d}\Sigma - \int_{\Omega'} \sigma'_{ij}(\overrightarrow{\Phi})\epsilon'_{ij}(\bar{\vec{v}}_0)\, \mathrm{d}\Omega' .$$

The integrals on S disappear since $\vec{v}_{0|S} = 0$ and the integrals on Σ disappear by virtue of (19). The variational formulation of the problem for $\vec{u}_0$ is to find $\vec{u}_0 \in V_0$ such that

$$\int_{\Omega'} \sigma'_{ij}(\vec{u}_0)\epsilon'_{ij}(\bar{\vec{v}}_0)\, \mathrm{d}\Omega' = -\int_{\Omega'} \sigma'_{ij}(\overrightarrow{\Phi})\epsilon'_{ij}(\bar{\vec{v}}_0)\, \mathrm{d}\Omega' \quad \forall \vec{v}_0 \in V_0 . \tag{20}$$

Conversely, let $\vec{u}_0$ a function of t with values in V_0 and verifying (20).

We have

$$\int_{\Omega'} \frac{\partial \sigma'_{ij}(\vec{u}_0)}{\partial x_j} \cdot \bar{v}_{0i}\, \mathrm{d}\Omega' = \int_{\Omega'} \left[\frac{\partial}{\partial x_j}[\sigma'_{ij}(\vec{u}_0)\bar{v}_{0i}] - \sigma'_{ij}(\vec{u}_0)\epsilon'_{ij}(\bar{\vec{v}}_0) \right] \mathrm{d}\Omega'$$

and an anlogous equation by replacing $\vec{u}_0$ by $\overrightarrow{\Phi}$.

Using (20), we obtain

$$-\int_{\Omega'} \frac{\partial \sigma'_{ij}(\vec{u}_0)}{\partial x_j} \cdot \bar{v}_{0i}\, \mathrm{d}\Omega' + \int_\Sigma \sigma'_{ij}(\vec{u}_0) n_{ej} \bar{v}_{0i}\, \mathrm{d}\Sigma = \int_{\Omega'} \frac{\partial \sigma'_{ij}(\overrightarrow{\Phi})}{\partial x_j} \cdot \bar{v}_{0i}\, \mathrm{d}\Omega' - \int_\Sigma \sigma'_{ij}(\overrightarrow{\Phi}) n_{ej} \bar{v}_{0i}\, \mathrm{d}\Sigma .$$

Taking $\vec{v} \in [\mathscr{D}(\Omega')]^3 \subset V_0$, we have

$$-\frac{\partial \sigma'_{ij}(\vec{u}_0)}{\partial x_j} = \frac{\partial \sigma'_{ij}(\overrightarrow{\Phi})}{\partial x_j} \quad \text{in} \quad \mathscr{D}(\Omega') .$$

Taking into account of $v_{0n|\Sigma} = 0$, we have

$$\int_{\Sigma} \overrightarrow{T}_t(\vec{u}_0) \cdot \vec{v}_{0t|\Sigma} \, \mathrm{d}\Sigma = - \int_{\Sigma} \overrightarrow{T}_t(\overrightarrow{\Phi}) \cdot \vec{v}_{0t|\Sigma} \, \mathrm{d}\Sigma \,,$$

and, since $\vec{v}_{0t|\Sigma}$ is arbitrary

$$\overrightarrow{T}_t(\vec{u}_0)_{|\Sigma} = - \overrightarrow{T}_t(\overrightarrow{\Phi})_{|\Sigma}$$

and we find the auxiliary problem.

Let us return to its variational formulation (20). The left-hand side can be considered as a scalar product in V_0:

$$\int_{\Omega'} \sigma'_{ij}(\vec{u}_0)\epsilon'_{ij}(\bar{\vec{v}}_0) \, \mathrm{d}\Omega' = (\vec{u}_0, \vec{v}_0)_{V_0} \ ,$$

The associated norm $\|\vec{u}_0\|_{V_0}$ being classically equivalent in V_0 to the norm $\|\vec{u}_0\|_1$ of $\Xi^1(\Omega')$. Since $\vec{u}_0 \in V_0 \subset \widehat{\Xi}^1(\Omega')$, we have

$$(\vec{u}_0, \vec{v}_0)_{V_0} = \int_{\Omega'} \sigma'_{ij}(\vec{u}_0)\epsilon'_{ij}(\bar{\vec{v}}_0) \, \mathrm{d}\Omega' = (\vec{u}_0, \vec{v}_0)_{\widehat{\Xi}^1(\Omega')} \ .$$

Setting $\vec{v}_0 = \vec{u}_0$, we have

$$\|\vec{u}_0\|_{V_0} = \|\vec{u}_0\|_{\widehat{\Xi}^1(\Omega')} \quad \forall \vec{u}_0 \in V_0 \ .$$

The variational Equation (20) can be written as

$$(\vec{u}_0, \vec{v}_0)_{V_0} = - \left(\overrightarrow{\Phi}, \vec{v}_0\right)_{\widehat{\Xi}^1(\Omega')} \quad \forall \vec{v}_0 \in V_0 \ . \tag{21}$$

But, we have

$$\left| \left(\overrightarrow{\Phi}, \vec{v}_0\right)_{\widehat{\Xi}^1(\Omega')} \right| \le \left\| \overrightarrow{\Phi} \right\|_{\widehat{\Xi}^1(\Omega')} \|\vec{v}_0\|_{V_0} \ ,$$

so that $- \left(\overrightarrow{\Phi}, \vec{v}_0\right)_{\widehat{\Xi}^1(\Omega')}$ is a continuous antilinear form on V_0.

Then, by the Lax- Milgram theorem, the precedent problem has one and only solution. Therefore, the problem (20) has one and one solution $\vec{u}_0 \in V_0$ and the auxiliary problem has one and only one solution $\vec{u}'$ in $\widehat{\Xi}^1(\Omega')$. The Equation (21) can be written

$$(\vec{u}', \vec{v}_0)_{\widehat{\Xi}^1(\Omega')} = 0 \quad \forall \vec{v}_0 \in V_0$$

and the solution $\vec{u}'$ of the auxiliary problem belongs to the orthogonal of V_0 in $\widehat{\Xi}^1(\Omega')$.

Step 3.

The solution $\vec{u}'$ of the auxiliary problem does not depend on $\overrightarrow{\Phi}$, since $\overrightarrow{\Phi}$ is not in the terms of the problem. We are going to use this remark for giving a estimate of $\|\vec{u}'\|_{\widehat{\Xi}^1(\Omega')}$.

We take, for $\overrightarrow{\Phi}$, a continuous lifting of $u_{n|\Sigma}\vec{n}$ in $\widehat{\Xi}^1(\Omega')$; we have

$$\left\| \overrightarrow{\Phi} \right\|_{\widehat{\Xi}^1(\Omega')} \le c \left\| u_{n|\Sigma} \right\|_{H^{1/2}(\Sigma)} \quad (c > 0) \ .$$

We have

$$\left| (\vec{u}_0, \vec{v}_0)_{V_0} \right| \le \left\| \overrightarrow{\Phi} \right\|_{\widehat{\Xi}^1(\Omega')} \|\vec{v}_0\|_{V_0}$$

and then

$$\|\vec{u}_0\|_{V_0} \le \left\| \overrightarrow{\Phi} \right\|_{\widehat{\Xi}^1(\Omega')}$$

and finally

$$\|\vec{u}_0\|_{V_0} \le c \left\| u_{n|\Sigma} \right\|_{H^{1/2}(\Sigma)}$$

For the solution $\vec{u}'$ of the auxiliary problem, we have

$$\vec{u}' = \vec{u}_0 + \overrightarrow{\Phi}$$

and then

$$\|\vec{u}'\|_{\widehat{\Xi}^1(\Omega')} \leq 2c \left\| u_{n|\Sigma} \right\|_{H^{1/2}(\Sigma)} . \tag{22}$$

Step 4.

Finally, we study $T_n(\vec{u}')_{|\Sigma}$ that is in the second dynamic condition (17) of the problem. We are going to show that it can be expressed by means of $u_{n|\Sigma}$. The solution $\vec{u}'$ of our problem verifies:

$$\frac{\partial \sigma'_{ij}(\vec{u}')}{\partial x_j} = 0 \quad \text{in} \quad \Omega' .$$

Let $\vec{w}'$ an element of $\widehat{\Xi}^1(\Omega')$. We have, by Green's formula and $\vec{w}'_{|S} = 0$:

$$0 = -\int_{\Omega'} \frac{\partial \sigma'_{ij}(\vec{u}')}{\partial x_j} \cdot \bar{w}_i \, \mathrm{d}\Omega' = -\int_\Sigma \sigma'_{ij}(\vec{u}') n_{\mathrm{ej}} \bar{w}_i \, \mathrm{d}\Sigma + \int_{\Omega'} \sigma'_{ij}(\vec{u}') \epsilon'_{ij}(\bar{\vec{w}}') \, \mathrm{d}\Omega' .$$

Since the solution $\vec{u}'$ of the initial problem verifies $\vec{T}_t(\vec{u}')_{|\Sigma} = 0$, the precedent equation gives:

$$\int_{\Omega'} \sigma'_{ij}(\vec{u}') \epsilon'_{ij}(\bar{\vec{w}}') \, \mathrm{d}\Omega' = -\int_\Sigma T_n(\vec{u}')_{|\Sigma} \, \bar{w}'_{n|\Sigma} \, \mathrm{d}\Sigma , \quad \forall \vec{w}' \in \widehat{\Xi}^1(\Omega') . \tag{23}$$

On the other hand, if $\vec{v}' \in [\mathscr{D}(\Omega')]^3$, we have

$$0 = -\left\langle \frac{\partial \sigma'_{ij}(\vec{u}')}{\partial x_j}, v'_i \right\rangle = \int_{\Omega'} \sigma'_{ij}(\vec{u}') \frac{\partial \bar{v}'_i}{\partial x_j} \, \mathrm{d}\Omega'$$

by virtue of the definition of the distributional derivatives. Therefore, we have

$$\int_{\Omega'} \sigma'_{ij}(\vec{u}') \epsilon'_{ij}(\bar{\vec{v}}') \, \mathrm{d}\Omega' = 0 \quad \forall \vec{v}' \in \left[\mathscr{D}(\Omega')\right]^3$$

and by density

$$\int_{\Omega'} \sigma'_{ij}(\vec{u}') \epsilon'_{ij}(\bar{\vec{v}}') \, \mathrm{d}\Omega' = 0 \quad \forall \vec{v}' \in \Xi^1(\Omega') .$$

Now, we are going to particularize $\vec{w}'$. Let call $w'_{n|\Sigma}$ a function defined on Σ and belonging to $H^{1/2}(\Sigma)$ and let take for $\vec{w}'$ a lifting of $w'_{n|\Sigma}\vec{n}$ in $\widehat{\Xi}^1(\Omega')$ (so that we have $\tilde{w}'_{n|\Sigma} = w'_{n|\Sigma}$). We set

$$\ell(\vec{w}') = \int_{\Omega'} \sigma'_{ij}(\vec{u}') \epsilon'_{ij}(\bar{\vec{w}}') \, \mathrm{d}\Omega' . \tag{24}$$

Since the difference between lifting belongs to $\Xi^1(\Omega')$, the right-hand side doesn't depend on the lifting $\vec{w}'$. Therefore, ℓ depends on $w'_{n|\Sigma}$. Let choose for $\vec{w}'$ a continuous lifting of $w'_{n|\Sigma}\vec{n}$; for this lifting, we have

$$\|\vec{w}'\|_{\widehat{\Xi}^1(\Omega')} \leq \alpha \left\| w'_{n|\Sigma}\vec{n} \right\|_{\left(H^{1/2}(\Sigma)\right)^3} , \quad (\alpha > 0)$$

and, if the components of $\vec{n}$ are sufficiently smooth:

$$\|\vec{w}'\|_{\widehat{\Xi}^1(\Omega')} \leq \beta \left\| w'_{n|\Sigma} \right\|_{H^{1/2}(\Sigma)} , \quad (\beta > 0)$$

But, we have

$$\left|\ell(\vec{w}')\right| \leq \|\vec{u}'\|_{\widehat{\Xi}^1(\Omega')} \cdot \|\vec{w}'\|_{\widehat{\Xi}^1(\Omega')}$$

and consequently

$$\left|\ell(\vec{\bar{w}}')\right| \leq \beta \left\|\vec{u}'\right\|_{\widehat{\Xi}^1(\Omega')} \cdot \left\|w'_{n|\Sigma}\right\|_{H^{1/2}(\Sigma)} \tag{25}$$

Then, since ℓ depends on $w'_{n|\Sigma}$, it is a continuous antilinear functional on $H^{1/2}(\Sigma)$, i.e an element of $\left[H^{1/2}(\Sigma)\right]'$. Taking into account of $\bar{w}'_{n|\Sigma} = w'_{n|\Sigma}$, the equation (23) can be written

$$\int_\Sigma T_n(\vec{u}')_{|\Sigma} \cdot \bar{w}'_{n|\Sigma} \, \mathrm{d}\Sigma = -\ell(\vec{\bar{w}}') \,,$$

so that the normal stress $T_n(\vec{u}')_{|\Sigma}$ can be considered as an element of $\left(H^{1/2}(\Sigma)\right)'$. Therefore, we have

$$\left|\left\langle T_n(\vec{u}')_{|\Sigma}, w'_{n|\Sigma}\right\rangle_{\left(H^{1/2}(\Sigma)\right)', H^{1/2}(\Sigma)}\right| \leq \beta \left\|\vec{u}'\right\|_{\widehat{\Xi}^1(\Omega')} \cdot \left\|w'_{n|\Sigma}\right\|_{H^{1/2}(\Sigma)}$$
$$\forall w'_{n|\Sigma} \in H^{1/2}(\Sigma) \,,$$

and then

$$\left\|T_n(\vec{u}')\right\|_{\left(H^{1/2}(\Sigma)\right)'} \leq \beta \left\|\vec{u}'\right\|_{\widehat{\Xi}^1(\Omega')} \,.$$

Using (22), we obtain finally

$$\left\|T_n(\vec{u}')\right\|_{\left(H^{1/2}(\Sigma)\right)'} \leq \delta \left\|u_{n|\Sigma}\right\|_{H^{1/2}(\Sigma)} \quad (\delta = 2c\beta) \,.$$

Consequently, there exists a continuous linear operator $\widehat{T}$ from $H^{1/2}(\Sigma)$ into $\left(H^{1/2}(\Sigma)\right)'$ such that

$$\widehat{T} u_{n|\Sigma} = -T_n(\vec{u}')_{|\Sigma} \,. \tag{26}$$

So, we have expressed linearly $T_n(\vec{u}')_{|\Sigma}$ by means of $u_{n|\Sigma}$. The linear operator $\widehat{T}$ depends on the elasticity of the body. We are going to prove that it has properties of symmetry and positivity. We introduce the analogous problem: to find $\vec{\tilde{u}}' \in \widehat{\Xi}^1(\Omega')$ verifying

$$-\frac{\partial \sigma'_{ij}(\vec{\tilde{u}}')}{\partial x_j} = 0 \text{ in } \Omega' \; ; \; \vec{\tilde{u}}'_{|S} = 0 \; ; \; \tilde{u}'_{n|\Sigma} = \tilde{u}_{n|\Sigma} \in H^{1/2}(\Sigma) \; ; \; \overrightarrow{T}_t\,(\vec{u}')_{|\Sigma} = 0 \,. \tag{27}$$

In (23), we replace $\vec{\bar{w}}'$ by $\vec{\tilde{u}}'$ and we have

$$\int_{\Omega'} \sigma'_{ij}(\vec{u}')\epsilon'_{ij}(\vec{\bar{\tilde{u}}}') \, \mathrm{d}\Omega' = -\int_\Sigma T_n(\vec{u}')_{|\Sigma} \, \bar{\tilde{u}}'_{n|\Sigma} \, \mathrm{d}\Sigma = \left\langle \widehat{T} u_{n|\Sigma}, \tilde{u}'_{n|\Sigma} \right\rangle$$

and since $\tilde{u}'_{n|\Sigma} = \tilde{u}_{n|\Sigma}$:

$$\int_{\Omega'} \sigma'_{ij}(\vec{u}')\epsilon'_{ij}(\vec{\bar{\tilde{u}}}') \, \mathrm{d}\Omega' = \left\langle \widehat{T} u_{n|\Sigma}, \tilde{u}_{n|\Sigma} \right\rangle \,.$$

Inverting roles of $\vec{u}'$ and $\vec{\tilde{u}}'$, we obtain

$$\int_{\Omega'} \sigma'_{ij}(\vec{\tilde{u}}')\epsilon'_{ij}(\vec{\bar{u}}') \, \mathrm{d}\Omega' = \left\langle \widehat{T} \tilde{u}_{n|\Sigma}, u_{n|\Sigma} \right\rangle \,.$$

By virtue of the classical symmetry of the left-hand side, we obtain the property of hermitian symmetry

$$\left\langle \widehat{T} u_{n|\Sigma}, \tilde{u}_{n|\Sigma} \right\rangle = \overline{\left\langle \widehat{T} \tilde{u}_{n|\Sigma}, u_{n|\Sigma} \right\rangle} \,.$$

Now, setting $\vec{\tilde{u}}' = \vec{u}'$, we have

$$\left\langle \widehat{T} u_{n|\Sigma}, u_{n|\Sigma} \right\rangle = \int_{\Omega'} \sigma'_{ij}(\vec{u}')\epsilon'_{ij}(\vec{\bar{u}}') \, \mathrm{d}\Omega' = \left\|\vec{u}'\right\|^2_{\widehat{\Xi}^1(\Omega')} \,.$$

By virtue of a trace theorem, we have

$$\left\|u_{n|\Sigma}\right\|_{H^{1/2}(\Sigma)} \leq C\left\|\vec{u}'\right\|_{\widehat{\Xi}^1(\Omega')} \quad (C>0)\,.$$

so that we have

$$\left\langle \widehat{T}u_{n|\Sigma}, u_{n|\Sigma}\right\rangle \geq C^{-2}\left\|u'_{n|\Sigma}\right\|^2_{H^{1/2}(\Sigma)}$$

and, since $u'_{n|\Sigma} = u_{n|\Sigma}$, the relation of positivity

$$\left\langle \widehat{T}u_{n|\Sigma}, u_{n|\Sigma}\right\rangle \geq C^{-2}\left\|u_{n|\Sigma}\right\|^2_{H^{1/2}(\Sigma)}\,.$$

The second dynamic condition (17) can be written as:

$$p_{|\Sigma} = \widehat{T}u_{n|\Sigma} + \rho_{0|_\Sigma} g\, n_{3|\Sigma}\, u_{n|\Sigma}\,. \tag{28}$$

So, we have reduced ou problem to a problem for a gas only:

$$\ddot{\vec{u}} = \overrightarrow{\text{grad}}\left(\frac{c_0^2(x_3)\text{div}\,[\rho_0(x_3)\vec{u}]}{\rho_0(x_3)}\right)\,. \tag{29}$$

$$-\,c_0^2\text{div}\,(\rho_0\vec{u})_{|\Sigma} = \widehat{T}u_{n|\Sigma} + \rho_{0|_\Sigma} g\, n_{3|\Sigma}\, u_{n|\Sigma}\,. \tag{30}$$

Afterwards, the auxiliary problem gives $\vec{u}'$, i.e. the motion of the elastic body.

5. Variational formulation of the problem

We consider a field of a admissible displacements $\bar{\vec{u}}(x_i)$, smooth in Ω and such that $\bar{\vec{u}} = \overrightarrow{\text{grad}}\,\tilde{\varphi}$. We have

$$\begin{aligned}\int_\Omega \rho_0\ddot{\vec{u}}\cdot\bar{\vec{u}}\,\text{d}\Omega &= \int_\Omega \rho_0\,\overrightarrow{\text{grad}}\left(\frac{c_0^2\text{div}\,(\rho_0\vec{u})}{\rho_0}\right)\cdot\bar{\vec{u}}\,\text{d}\Omega\\ &= \int_\Sigma c_0^2\text{div}\,(\rho_0\vec{u})\cdot\bar{u}_{n|\Sigma}\,\text{d}\Sigma - \int_\Omega \frac{c_0^2}{\rho_0}\text{div}\,(\rho_0\vec{u})\,\text{div}\left(\rho_0\bar{\vec{u}}\right)\,\text{d}\Omega\end{aligned}$$

and then

$$\left.\begin{aligned}\int_\Omega \rho_0\ddot{\vec{u}}\cdot\bar{\vec{u}}\,\text{d}\Omega + \int_\Omega \frac{c_0^2}{\rho_0}\text{div}\,(\rho_0\vec{u})\,\text{div}\left(\rho_0\bar{\vec{u}}\right)\,\text{d}\Omega\\ +\int_\Sigma\left(\widehat{T}u_{n|\Sigma} + \rho_{0|_\Sigma} g\, n_{3|\Sigma}\, u_{n|\Sigma}\right)\bar{u}_{n|\Sigma}\,\text{d}\Sigma = 0\,.\end{aligned}\right\} \tag{31}$$

Conversely, let $\vec{u}$ a function of t with values in the field of the admissible displacements and verifying (31). We obtain easily from (31)

$$\left.\begin{aligned}0 = \int_\Omega \rho_0\left[\ddot{\vec{u}} - \overrightarrow{\text{grad}}\left(\frac{c_0^2}{\rho_0}\text{div}\,(\rho_0\vec{u})\right)\right]\cdot\overrightarrow{\text{grad}}\,\bar{\tilde{\varphi}}\,\text{d}\Omega\\ +\int_\Sigma\left[c_0^2\text{div}\,(\rho_0\vec{u}) + \widehat{T}u_{n|\Sigma} + \rho_{0|_\Sigma} g\, n_{3|\Sigma}\, u_{n|\Sigma}\right]\frac{\partial\bar{\tilde{\varphi}}}{\partial n}_{|\Sigma}\,\text{d}\Sigma\end{aligned}\right\}\ \forall\bar{\vec{u}} = \overrightarrow{\text{grad}}\,\tilde{\varphi}\,.$$

or

$$\left.\begin{aligned}0 = \int_\Sigma \bar{\tilde{\varphi}}\cdot\rho_0\left[\ddot{\vec{u}} - \overrightarrow{\text{grad}}\left(\frac{c_0^2}{\rho_0}\text{div}\,(\rho_0\vec{u})\right)\right]\cdot\vec{n}_{|\Sigma}\,\text{d}\Sigma\\ -\int_\Sigma\left[c_0^2\text{div}\,(\rho_0\vec{u}) + \widehat{T}u_{n|\Sigma} + \rho_{0|_\Sigma} g\, n_{3|\Sigma}\, u_{n|\Sigma}\right]\frac{\partial\bar{\tilde{\varphi}}}{\partial n}_{|\Sigma}\,\text{d}\Sigma\\ -\int_\Omega \text{div}\left[\rho_0\left(\ddot{\vec{u}} - \overrightarrow{\text{grad}}\left(\frac{c_0^2}{\rho_0}\text{div}\,(\rho_0\vec{u})\right)\right)\right]\cdot\bar{\tilde{\varphi}}\,\text{d}\Omega\,.\end{aligned}\right\} \tag{32}$$

Taking $\tilde{\varphi} \in \mathscr{D}(\Omega)$ and setting

$$\overrightarrow{\Phi}_0 = \rho_0 \, \overrightarrow{\text{grad}} \left[\ddot{\varphi} - \frac{c_0^2}{\rho_0} \text{div} \left(\rho_0 \, \overrightarrow{\text{grad}} \, \varphi \right) \right] ,$$

we have

$$\text{div} \, \overrightarrow{\Phi}_0 = 0 \quad \text{in} \quad \Omega \, . \tag{33}$$

Taking $\tilde{\varphi}_{|\Sigma}$ arbitrary and $\frac{\partial \tilde{\varphi}}{\partial n}_{|\Sigma} = 0$, we obtain

$$\rho_0 \left[\ddot{\vec{u}} - \overrightarrow{\text{grad}} \left(\frac{c_0^2}{\rho_0} \text{div} \left(\rho_0 \vec{u} \right) \right) \right] \cdot \vec{n} = 0 \quad \text{on} \quad \Sigma$$

or

$$\overrightarrow{\Phi}_0 \cdot \vec{n} = 0 \quad \text{on} \quad \Sigma \, . \tag{34}$$

Finally, taking $\frac{\partial \tilde{\varphi}}{\partial n}_{|\Sigma}$ arbitrary, we have

$$c_0^2 \text{div} \left(\rho_0 \vec{u} \right)_{|\Sigma} + \widehat{T} u_{n|\Sigma} + \rho_{0|_\Sigma} g \, n_{3|\Sigma} \, u_{n|\Sigma} = 0 \, ,$$

i.e the dynamic condition (30). Since

$$\overrightarrow{\Phi}_0 = \rho_0 \, \overrightarrow{\text{grad}} \, \Psi \, , \quad \text{with} \quad \Psi = \ddot{\varphi} - \frac{c_0^2}{\rho_0} \, \text{div} \left(\rho_0 \overrightarrow{\text{grad}} \, \varphi \right) \, ,$$

the Equations (33) and (34) give

$$\text{div} \left(\rho_0 \, \overrightarrow{\text{grad}} \Psi \right) = 0 \quad \text{in} \quad \Omega \, ; \quad \frac{\partial \Psi}{\partial n}_{|\Sigma} = 0 \, . \tag{35}$$

This Weumann problem has for solution only $\Psi = constant$ and consequently

$$\ddot{\vec{u}} - \overrightarrow{\text{grad}} \left(\frac{c_0^2}{\rho_0} \text{div} \left(\rho_0 \vec{u} \right) \right) = 0 \, .$$

6. The problem is a classical vibration problem

Step 1.

We precise the field of the admissible displacements by introducing the space V:

$$V = \left\{ \begin{array}{l} \vec{u} \in \mathscr{L}^2(\Omega) \stackrel{\text{def}}{=} \left[L^2(\Omega) \right]^3 \; ; \; \vec{u} = \overrightarrow{\text{grad}} \, \tilde{\varphi} \; ; \; \tilde{\varphi} \in \widetilde{H}^1(\Omega) \; ; \; \text{div} \left(\rho_0 \vec{\tilde{u}} \right) \in L^2(\Omega); \\ \frac{\partial \tilde{\varphi}}{\partial n}_{|\Sigma} = \tilde{u}_{n|\Sigma} \in H^{1/2}(\Sigma) \, . \end{array} \right\} ,$$

equipped with the hilbertian norm defined by

$$\|\vec{u}\|_V^2 = \int_\Omega \rho_0 \, |\vec{u}|^2 \, d\Omega + \int_\Omega |\text{div} \left(\rho_0 \vec{u} \right)|^2 \, d\Omega + \left\| u_{n|\Sigma} \right\|_{H^{1/2}(\Sigma)}^2 ,$$

and the space H completion of V for the norm associated to the scalar product

$$\left(\vec{u}, \vec{\tilde{u}} \right)_H = \int_\Omega \rho_0 \vec{u} \cdot \bar{\vec{\tilde{u}}} \, d\Omega \, .$$

Setting

$$a \left(\vec{u}, \vec{\tilde{u}} \right) = \int_\Omega \frac{c_0^2}{\rho_0} \text{div} \left(\rho_0 \vec{u} \right) \text{div} \left(\rho_0 \bar{\vec{\tilde{u}}} \right) \, d\Omega + \int_\Sigma \left(\widehat{T} u_{n|\Sigma} + \rho_{0|_\Sigma} g \, n_{3|\Sigma} \, u_{n|\Sigma} \right) \bar{\tilde{u}}_{n|\Sigma} \, d\Sigma \, ,$$

we obtain the precise variational formulation of the problem. To find $\vec{u}(\cdot) \in V$ such that

$$\left(\ddot{\vec{u}}, \tilde{\vec{u}}\right)_H + a\left(\vec{u}, \tilde{\vec{u}}\right) = 0 \quad \forall \tilde{\vec{u}} \in V\,.$$

Step 2.

Let us study the hermitian sesquilinear form

$$\mathscr{C}\left(u_{n|\Sigma}, \tilde{u}_{n|\Sigma}\right) \stackrel{\text{def}}{=} \int_\Sigma \left(\widehat{T} u_{n|\Sigma} + \rho_{0|\Sigma} g\, n_{3|\Sigma}\, u_{n|\Sigma}\right) \bar{\tilde{u}}_{n|\Sigma}\, \mathrm{d}\Sigma$$

$\mathscr{C}$ is continuous on $H^{1/2}(\Sigma) \times H^{1/2}(\Sigma)$ and we have:

$$\mathscr{C}\left(u_{n|\Sigma}, u_{n|\Sigma}\right) \geq \left(C^{-2} - \max\rho_{0|\Sigma}\, g\right) \left\|u_{n|\Sigma}\right\|^2_{H^{1/2}(\Sigma)}\,.$$

In the following, we suppose that $\mathscr{C}$ is coercive, i.e

$$C^{-2} - \max\rho_{0|\Sigma}\, g > 0$$

(for example, if $\max\rho_{0|\Sigma}$ is sufficiently small).

Then, $\left[\mathscr{C}\left(u_{n|\Sigma}, u_{n|\Sigma}\right)\right]^{1/2}$ defines on $H^{1/2}(\Sigma)$ a norm that is equivalent to the classical norm of $H^{1/2}(\Sigma)$.

Step 3.

In order to prove that the problem is a classical vibration problem, we use the method that is introduced in [7]. We must prove that

a) $\left[a\left(\vec{u}, \vec{u}\right)\right]^{1/2}$ defines on V a norm equivalent to $\|\vec{u}\|_V$.

b) The imbedding $V \subset H$, obviously dense and continuous, hence compact. We omit the proof that is strictly identical to the proof in [7], p66-68. Therefore there exists a denumerable infinity of positive real eigenvalues ω_p^2:

$$0 < \omega_1^2 \leq \omega_2^2 \leq \cdots \leq \omega_p^2 \leq \cdots\,;\ \omega_p^2 \to +\infty \text{ when } p \to +\infty\,.$$

The eigenelements $\left\{\vec{u}_p\right\}$ form an orthonormal basis in H and an orthogonal basis in V equipped with the scalar product $\left(\vec{u}, \tilde{\vec{u}}\right)_V$.

To each eigenmotion $\left\{\vec{u}_p\right\}$ of the gas corresponds an eigenmotion $\left\{\vec{u}'_p\right\}$ of the elastic body verifying

$$\left\|\vec{u}'_p\right\|_{\widehat{\Xi}^1(\Omega')} \leq 2c \left\|u_{np|\Sigma}\right\|_{H^{1/2}(\Sigma)}\,.$$

■.

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